Problem 5191. Let a, b, c be positive real numbers such that $ab+bc+ca=3$. Prove that

$$\frac{a\sqrt{bc}+b\sqrt{ca}+c\sqrt{ab}}{a^4+b^4+c^4}\le 1$$

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According to AM-GM and Power Mean inequalities we get

$$a\sqrt{bc}+b\sqrt{ca}+c\sqrt{ab}\le a\frac{b+c}{2}+b\frac{c+a}{2}+c\frac{a+b}{2}=ab+bc+ca \qquad (1)$$

$$\sqrt[4]{\frac{a^4+b^4+c^4}{3}}\ge\sqrt{\frac{a^2+b^2+c^2}{3}} \quad\Leftrightarrow\quad a^4+b^4+c^4\ge\frac{\left(a^2+b^2+c^2\right)^2}{3} \qquad (2)$$

By (1) and (2), using the well know inequality $a^2+b^2+c^2\ge ab+bc+ca$, we obtain

$$\begin{aligned}\frac{a\sqrt{bc}+b\sqrt{ca}+c\sqrt{ab}}{a^4+b^4+c^4} &\le \frac{ab+bc+ca}{\frac{1}{3}\left(a^2+b^2+c^2\right)^2}\\ &\le \frac{3(ab+bc+ca)}{(ab+bc+ca)^2}\\ &= \frac{3}{ab+bc+ca}=1\end{aligned}$$

This ends the proof. Clearly, equality occurs for $a=b=c$. □